Scaling Technique in Tempered Stable Processes and Its Application to Financial Data Analysis

Liuming Wang[†], Kirill Klimov[†], Minseob Kim[*], Yiming Chen[*], Dorian Zhou[⋆], Young Shin Kim[†,‡]

[†] Applied mathematics and Statistics, Stony Brook University
[*] Ward Melville High School
[⋆] R.C. Murphy Junior High School
‡ Faculty Supervisor. College of Business, Stony Brook University

February 23, 2020

Abstract

https://aaronga.shinyapps.io/TSFit_AA/
https://aaronga.shinyapps.io/NTSDistFigureOut/

Normal Tempered Stable Distribution

Let $\alpha \in (0, 2)$, $\theta, \gamma > 0$, and $\mu, \beta \in \mathbb{R}$. Let $\mathcal{T}$ be a positive random variable whose characteristic function $\phi_{\mathcal{T}}$ is equal to

$$\phi_{\mathcal{T}}(u) = \exp\left(-\frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left((\theta - iu)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)\right). \tag{1}$$

The random variable $\mathcal{T}$ is referred to as *Tempered Stable Subordinator*. The *normal tempered stable* (NTS) random variable X with parameters $(\alpha, \theta, \beta, \gamma, \mu)$ is defined as

$$X = \mu - \beta + \beta\mathcal{T} + \gamma\sqrt{\mathcal{T}}W, \tag{2}$$

where $W \sim N(0, 1)$ is independent of $\mathcal{T}$, and we denote $X \sim \mathrm{NTS}(\alpha, \theta, \beta, \gamma, \mu)$. The characteristic function (Ch.F) of ϵ is given by

$$\begin{aligned}\phi_{NTS}(u) &= E[e^{iuX}] \\ &= \exp\left((\mu - \beta)iu - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left(\left(\theta - i\beta u + \frac{\gamma^2 u^2}{2}\right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)\right).\end{aligned}$$

Normal Tempered Stable Distribution

The first four moments of X are as follows:

- Mean: $E[X] = \mu$
- Variance: $\mathrm{var}(X) = \gamma^2 + \beta^2 \left(\frac{2-\alpha}{2\theta} \right)$
- skewness: $\mathrm{S}(X) = \frac{\beta\,(2-\alpha)\,\left(6\,\gamma^2\,\theta - \alpha\beta^2 + 4\beta^2\right)}{\sqrt{2\theta}\,(2\,\gamma^2\,\theta - \alpha\beta^2 + 2\beta^2)^{3/2}}$
- Excess kurtosis: $\mathrm{K}(X) =$
$\frac{(2-\alpha)\,\left(\alpha^2\beta^4 - 10\,\alpha\beta^4 - 12\,\alpha\beta^2\,\gamma^2\,\theta + 24\beta^4 + 48\beta^2\,\gamma^2\,\theta + 12\,\gamma^4\,\theta^2\right)}{2\,\theta\,(2\,\gamma^2\,\theta - \alpha\beta^2 + 2\beta^2)^2}$

Normal Tempered Stable Process

The NTS distribution is purely non-Gaussian infinitely divisible, we can define a pure jump Lévy process $(X_t)_{t\geq 0}$ such that $X_1 \sim \text{NTS}(\alpha, \theta, \beta, \gamma, m)$. In this case, we say that $(X_t)_{t\geq 0}$ is NTS process with parameters $(\alpha, \theta, \beta, \gamma, m)$. The ch.F of X_t is

$$\phi_{X_t}(u) = \exp(t \log(\phi_{NTS}(u; \alpha, \theta, \beta, \gamma, \mu))).$$

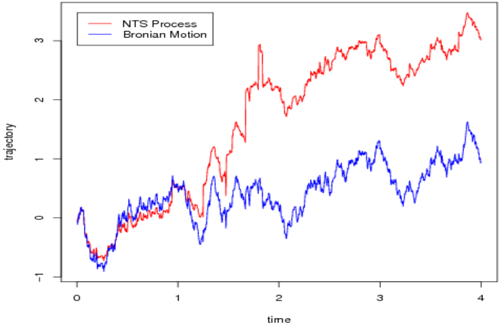
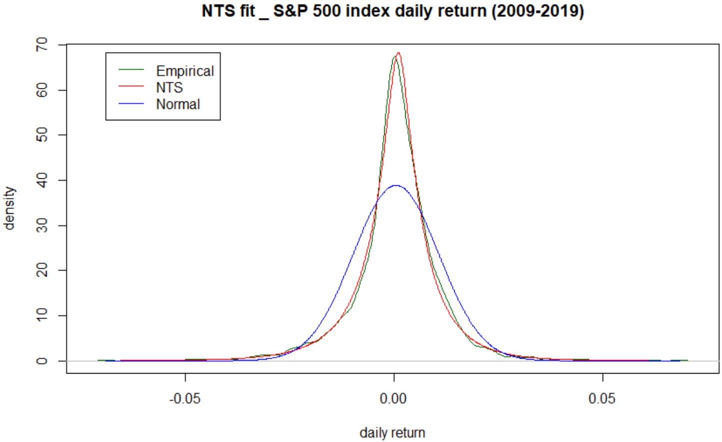


Figure: NTS Process is more volatile than Brownian Motion ($\mu = -0.01, \sigma = 1$)

S&P 500 return distribution

- Fattails
- Leptokurtic distributed
- Skewed left



stdNTS Distribution

If $\mu = 0$ and $\gamma = \sqrt{1 - \beta^2 \left(\frac{2-\alpha}{2\theta}\right)}$ with $|\beta| < \sqrt{\frac{2\theta}{2-\alpha}}$ then $\epsilon \sim \mathrm{NTS}(\alpha, \theta, \beta, \gamma, \mu)$ has zero mean and unit variance.
Put $\beta = B\sqrt{\frac{2\theta}{2-\alpha}}$ for $B \in (-1, 1)$, then $|\beta| < \sqrt{\frac{2\theta}{2-\alpha}}$ and $\gamma = \sqrt{1 - B^2}$.
Then the Ch.F of ϵ equals to

$$\phi_{stdNTS}(u; \alpha, \theta; B) = \phi_\epsilon(u) = \phi_\epsilon(u) = E[e^{iu\epsilon}]$$

$$= \exp\left(-iuB\sqrt{\frac{2\theta}{2-\alpha}} - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left(\left(\theta - iuB\sqrt{\frac{2\theta}{2-\alpha}} + \frac{u^2}{2}\left(1 - B^2\right)\right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)\right)$$

In this case ϵ is referred to as the *standard NTS* random variable with parameters $(\alpha, \theta; B)$, and we denote $\epsilon \sim \mathrm{stdNTS}(\alpha, \theta; B)$.

stdNTS Distribution

For $\epsilon \sim \text{stdNTS}(\alpha, \theta; B)$, we have

$$S(\epsilon) = \sqrt{\frac{2-\alpha}{2\theta}} B \left(3(1-B^2) + \frac{4-\alpha}{2-\alpha} B^2 \right) \tag{3}$$

and

$$K(\epsilon) = \frac{(2-\alpha)}{2\theta} \left((\alpha-4)(\alpha-6) \left(\frac{B^2}{2-\alpha} \right)^2 + \left((24-6\alpha) \left(\frac{B^2}{2-\alpha} \right) + 3(1-B^2) \right) \right) \tag{4}$$

Scaling Technique - Normal Distribution

Normal Distribution Case: General normal distribution is obtained by the standard normal distribution.

$$X = \mu + \sigma\epsilon$$

where $X \sim N(\mu, \sigma)$ and $\epsilon \sim N(0, 1)$. And we have

$$F_X(x) = F_\epsilon\left(\frac{x-\mu}{\sigma}\right), \quad f_X(x) = \frac{1}{\sigma} f_\epsilon\left(\frac{x-\mu}{\sigma}\right)$$

where F_X and F_ϵ are CDF of X and ϵ respectively, and f_X and f_ϵ are PDF X and of ϵ, respectively.

Scaling Technique - Brownian Motion

Brownian Motion (BM) Case: Let $\{W_t\}_{t \geq 0}$ be the BM, and consider an arithmetic BM $\{X_t\}_{t \geq 0}$ with

$$dX_t = \mu\, dt + \sigma dW_t.$$

Then we have

$$X_{\Delta t} = X_{t+\Delta t} - X_{\Delta t} = \mu \Delta t + \sigma W_{\Delta t} = \mu \Delta t + \sigma \epsilon \sqrt{\Delta t}.$$

Hence

$$F_{X_{\Delta t}}(x) = F_{\epsilon}\left(\frac{x - \mu \Delta t}{\sigma \sqrt{\Delta t}}\right), \quad f_{X_{\Delta t}}(x) = \frac{1}{\sigma \sqrt{\Delta t}} f_{\epsilon}\left(\frac{x - \mu \Delta t}{\sigma \sqrt{\Delta t}}\right)$$

Scaling Technique - NTS Process

Lemma: Let $(X_t)_{t\geq 0}$ be a NTS process with parameters $(\alpha, \theta, \beta, \gamma, m)$, and $X_{\Delta t} = X_{t+\Delta t} - X_t$. Let $s = \sqrt{\gamma^2 + \beta^2\left(\frac{2-\alpha}{2\theta}\right)}$. Suppose $\xi \sim \text{stdNTS}(\bar{\alpha}, \bar{\theta}, \bar{\beta})$ where $\bar{\alpha} = \alpha$, $\bar{\theta} = \theta\Delta t$, $\bar{\beta} = \beta\sqrt{\Delta t}/s$. Then we have $X_{\Delta t} \overset{\mathrm{d}}{=} \mu + \sigma\xi$ where $\sigma = s\sqrt{\Delta t}$, and $\mu = m\Delta t$.

Apply the Lemma, we have

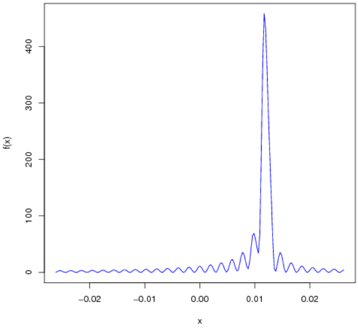
$$F_{X_{\Delta t}}(x) = F_\xi\left(\frac{x-\mu}{\sigma}\right)$$

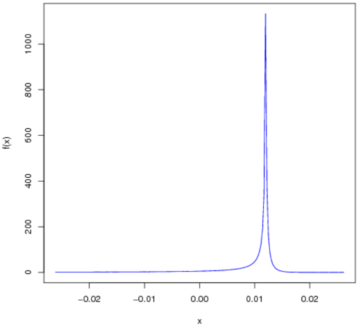
where $F_{X_{\Delta t}}$ and F_ξ are the CDF of $X_{\Delta t}$ and ξ, respectively. Moreover, we have

$$f_{X_{\Delta t}}(x) = \frac{1}{\sigma} f_\xi\left(\frac{x-\mu}{\sigma}\right)$$

where $f_{X_{\Delta t}}$ and f_ξ are the PDF of $X_{\Delta t}$ and ξ, respectively.

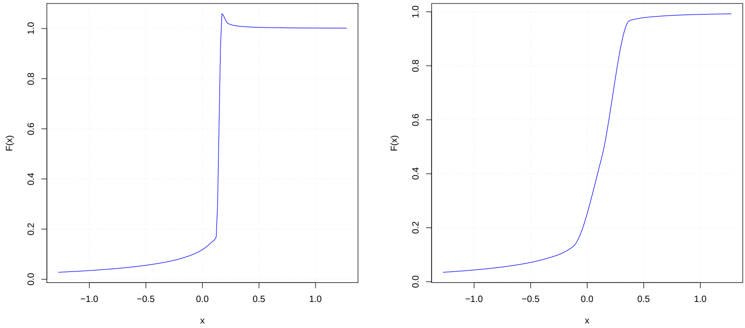
Scaling Technique - PDF





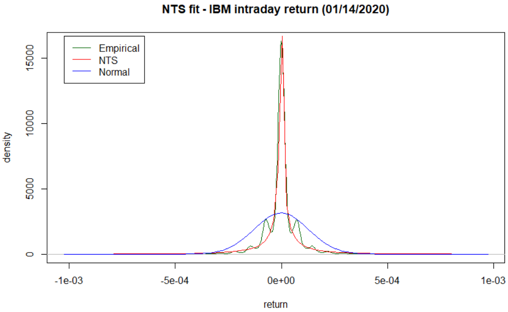
PDF's of NTS distribution with parameters $\alpha = 0.5$, $\theta = 10$, $\beta = -3$, $\gamma = 0.1$, $m = 0$, and $\Delta t = 1/250$ (one business day). The scaling technique is not used in the left plate while it is used in the right plate.

Scaling Technique - CDF



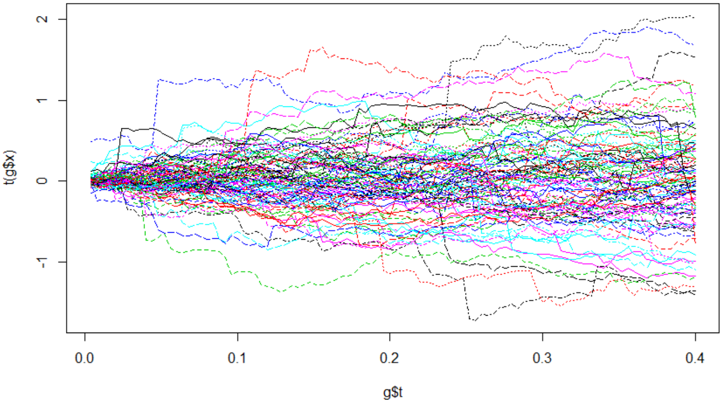
CDF's of NTS distribution with parameters $\alpha = 0.2$, $\theta = 1$, $\beta = -3$, $\gamma = 0.1$, $m = 0$, and $\Delta t = 0.05$. The scaling technique is not used in the left plate while it is used in the right plate.

Scaling Technique - Intraday Trading Data Fit



IBM Intraday trading data fit.
Estimated parameters $\alpha = 0.3767$, $\theta = 0.00783$, $\beta = 1.307 \cdot 10^{-7}$, $\gamma = 1.2675 \cdot 10^{-4}$, $m = -1.6109 \cdot 10^{-7}$
Goodness of fit test: KS statistic = 0.15 (p-value = 0.2105)

Scaling Technique - Sample Path Generation



Parameter Fit : DJIA 30 members

Fit normal parameters (μ, σ) to the daily log-return of 30 stocks in DJIA members.

Ticker	μ	σ	KS	p-value	Ticker	μ	σ	KS	p-value
V	0.0010	0.0160	0.0683	0.00%	CSCO	0.0005	0.0165	0.0936	0.00%
MMM	0.0005	0.0131	0.0885	0.00%	AXP	0.0007	0.0196	0.1164	0.00%
AAPL	0.0012	0.0167	0.0658	0.00%	BA	0.0008	0.0167	0.0607	0.01%
CAT	0.0005	0.0191	0.0741	0.00%	CVX	0.0003	0.0138	0.0589	0.01%
KO	0.0005	0.0099	0.0622	0.00%	DD	0.0005	0.0217	0.0831	0.00%
XOM	0.0001	0.0122	0.0564	0.03%	GS	0.0004	0.0194	0.0795	0.00%
HD	0.0009	0.0137	0.0672	0.00%	IBM	0.0003	0.0129	0.0752	0.00%
INTC	0.0006	0.0163	0.0535	0.07%	JNJ	0.0004	0.0097	0.0723	0.00%
JPM	0.0006	0.0213	0.1113	0.00%	MCD	0.0005	0.0103	0.0593	0.01%
MRK	0.0005	0.0133	0.0661	0.00%	MSFT	0.0008	0.0153	0.0712	0.00%
NKE	0.0009	0.0157	0.0665	0.00%	PFE	0.0004	0.0125	0.0651	0.00%
PG	0.0004	0.0100	0.0701	0.00%	TRV	0.0005	0.0129	0.0741	0.00%
UNH	0.0009	0.0168	0.0698	0.00%	UTX	0.0005	0.0132	0.0723	0.00%
VZ	0.0004	0.0111	0.0455	0.64%	WMT	0.0004	0.0112	0.0687	0.00%
WBA	0.0004	0.0163	0.0701	0.00%	DIS	0.0007	0.0147	0.0734	0.00%